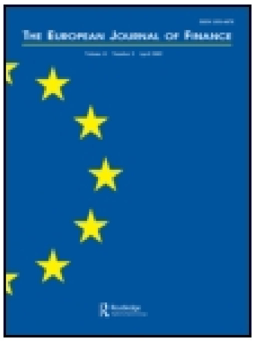

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Optimal allotment policy in central bank open market operations[1]

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This article derives a central bank's optimal liquidity supply towards a money market with an unrestricted lending facility. We show that when the effect of liquidity on market rates is not too small, and the monetary authority is concerned with both interest rates and liquidity conditions, then the optimal allotment policy may entail a 'discontinuous' reaction to initial conditions. In particular, the model predicts a threshold level of liquidity below which the central bank will not bail out the banking system. An estimation of the liquidity effect for the euro area suggests that the discontinuity might have contributed to the Eurosystem's tight response to occurrences of underbidding during the period June 2000 through March 2004.

Keywords: open market operations; liquidity effect; standing facilities; underbidding

JEL codes: E43; E52

1. Introduction

The topic of monetary policy implementation, i.e., the issue of how a central bank implements its monetary decisions, has recently attracted significant interest from both central bankers and academic researchers. The aim of this article is to contribute to the understanding of monetary policy implementation, especially concerning the mechanics that link the decisions of the monetary authority and the behavior of short-term interest rates in the interbank market. To this aim, we propose a simple model that captures some of the institutional aspects of present-day operational frameworks, where we focus on the active role of the central bank as a provider of liquidity to the banking system.

Our analytic framework builds upon Woodford's (2003) basic model according to which the central bank determines the level of market interest rates essentially by positioning an interest rate corridor the upper boundary of which is defined through the rate applied in a standing leading facility. Changes in policy rates are implemented by moving the corridor up or down. Within the interest rate corridor, supposing it is sufficiently wide as in the cases of the Federal Reserve or the Eurosystem, there is some scope for market rates to vary on a daily basis. Within the corridor, interest rates may be steered indirectly by providing more or less liquidity to the market. Changes to the market rate result then, in particular, from the so-called end-of-period liquidity effect (Hamilton 1997), which summarizes the response of the market to expectations concerning tighter or looser liquidity conditions at the end of the reserve accounting period. For example, if the rate for overnight interbank loans appears to be too high within the corridor from a central

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bank perspective, a liquidity injection would bring a certain relief to market conditions and would allow market rates to decline again to a more desirable level.

A central assumption made in this article is that the central bank wishes to implement its interest rate target within the corridor in a *smooth manner*. By this formulation, we mean that the monetary authority will have a preference for keeping fluctuations, both in the market rate and in the central bank credit outstanding to the banking sector, at a minimum. This can be a plausible assumption, in particular if the liquidity provided by the central bank in one reserve maintenance period contributes to the fulfillment of reserve requirements in another period, and if the central bank wishes to sustain the effectiveness of its regular instruments. For example, in the case of the operational framework of the Eurosystem in use before March 2004, a large allotment in the last main refinancing operation of a reserve maintenance period would imply the subsequent period starting with an excessively large central bank credit. This situation, possibly deteriorated by a liquidity-providing shock, would dwarf the first main refinancing operation in the new period, meaning that the regular instruments of implementation would temporarily cease to ascertain an effective control over liquidity conditions in the interbank market.

Our results are as follows. In the absence of imbalances, there is no trade-off between the criteria of interest rate and liquidity smoothing. Indeed, in this case, the central bank may simply inject the aggregate reserve deficit and may thereby reach both interest rate and liquidity targets. However, typically, this will not be feasible. Instead, there will be a trade-off between interest rate and liquidity smoothing after an occurrence of *underbidding* in an individual open market operation. Here and elsewhere, the term underbidding refers to a situation where the total of the incoming bids in a central bank operation is lower than the allotment that would be necessary to implement the central bank's regular liquidity policy. Because in any given operation, the central bank can only allocate liquidity by satisfying incoming bids, the allotment is bound to be below the necessary amount in the case of underbidding, so that a temporary shortage of liquidity in the money market will be created. Bailing out the banking system would require an overly large allotment in the subsequent open market operation, which causes the trade-off between interest rate and liquidity smoothing.

To our knowledge, the issue of an optimal policy reaction to underbidding has not been addressed so far in the literature. Most of the established theoretical literature on the interbank market for reserves in times of regular operation, as originated by the work of Poole (1968), Ho and Saunders (1985), Campbell (1987), Spindt and Hoffmeister (1988), Hamilton (1996), and others, has tended to abstract from the central bank's active role in the money market, with only a few recent exceptions.

Ayuso and Repullo (2003) assume that the central bank's loss function penalizes market rates below the target more severely than market rates above the target. This induces the central bank to follow a tight allotment policy. As a consequence, allotments in fixed-rate tenders (where banks pay the target rate) are profitable, generating overbidding. However, in a variable-rate tender with pre-announced liquidity injection, there is a bidding equilibrium without excess demand. Bartolini, Bertola, and Prati (2002) develop an intertemporal model of the market for Federal funds allowing for daily central bank intervention. Without intervention, the expected variance of the market rate is increasing over the reserve accounting period. With unconstrained intervention, however, the central bank may implement its target interest rate on all but the final day of the reserve accounting period. The model thereby allows a positive analysis of the consequences of various central bank policies on the time-series properties of the Federal funds rate. Bartolini, Bertola, and Prati (2001) offer an explanation for the empirical observation that banks in the U.S. tend to hold more reserves on settlement days than on other days of the reserve accounting period.

It is shown that with uncertain reserve requirements, transaction costs, and constant market rates, demand for reserves is higher on the second day of a two-day period than on the first day. While the paper does not specify an objective function for the central bank, it discusses informally the trade-off between a higher interest rate vis-à-vis increased reserves on settlement day from the central bank's perspective.

The rest of this article is structured as follows. In Section 2, we describe the indirect steering of interest rates via the liquidity effect and derive the liquidity target from the interest rate target. In Section 3, we consider allotment decisions in two consecutive open market operations and derive the benchmark allotment in the first operation that allows satisfying both the interest rate and the liquidity target in the second operation. Section 4 analyzes the optimal policy reaction to underbidding and its responsiveness to changes in the initial conditions. In Section 5, we quantify the liquidity effect in the euro area and evaluate the practical relevance of the identified discontinuity. Section 6 discusses robustness and the case of excess liquidity after the first operation. Section 7 concludes. Appendix 1 contains the proof of Proposition 4. Ewerhart et al. (2007) document and model manipulation of the euro money market. In the formal analysis, a commercial bank finds it in its best interest to sporadically leverage its exposure to the interbank rate by contracting in the market for overnight index swaps. The width of the corridor is shown to be of no consequence for either probability or extent of manipulation. Instead, fine-tuning operations and careful design of the operational framework can mitigate the problem.

2. Steering interest rates via the liquidity effect

We consider a single reserve accounting or maintenance period represented by the interval $[0; T]$, where $T > 0$. The expected aggregate liquidity supply provided through central bank operations over the maintenance period will be denoted by S. Aggregate liquidity demand is equal to the sum of exogenous aggregate reserve requirements $\bar{R}$ and stochastic autonomous liquidity factors $\tilde{A}$. Liquidity demand is assumed to be perfectly inelastic. Autonomous liquidity factors realize at some time t_{s}, where $0 < t_{\mathrm{s}} < T$. We will denote by $G(.)$ the cumulative distribution function corresponding to $\tilde{A}$. Figure 1 shows the empirical distribution of autonomous factors in the euro area.

Standing facilities offer individual banks the opportunity to borrow and lend an arbitrary amount overnight at the marginal lending rate r^{L} and the deposit rate $r^{\mathrm{D}} < r^{\mathrm{L}}$, respectively. Clearly, in the absence of a deposit facility, we have $r^{\mathrm{D}} = 0$. From the inelasticity of demand at the end of the reserve maintenance period and from the availability of the standing facilities, it follows that the market rate $r^s(S, \tilde{A})$ prevailing after the realization of the autonomous factor shock $\tilde{A}$ is equal to one of the corridor rates, depending on whether there is excess demand or excess supply at the end of that given period. Specifically, we have

$$r^s(S, \tilde{A}) = \begin{cases} r^{\mathrm{L}} & \text{if } S < \bar{R} + \tilde{A} \\ r^{\mathrm{D}} & \text{if } S \geq \bar{R} + \tilde{A} \end{cases}.$$

Thus, in the model the market rate reaches the marginal lending rate when the aggregate average liquidity supply is below aggregate demand, and analogously the market rate drops to the deposit rate when supply exceeds demand. Hence, with reference to the rational expectations hypothesis, the level of the market rate within the period can be expressed as a weighted average of the rates of the standing facilities, where the weights are given by the respective probabilities that the upper

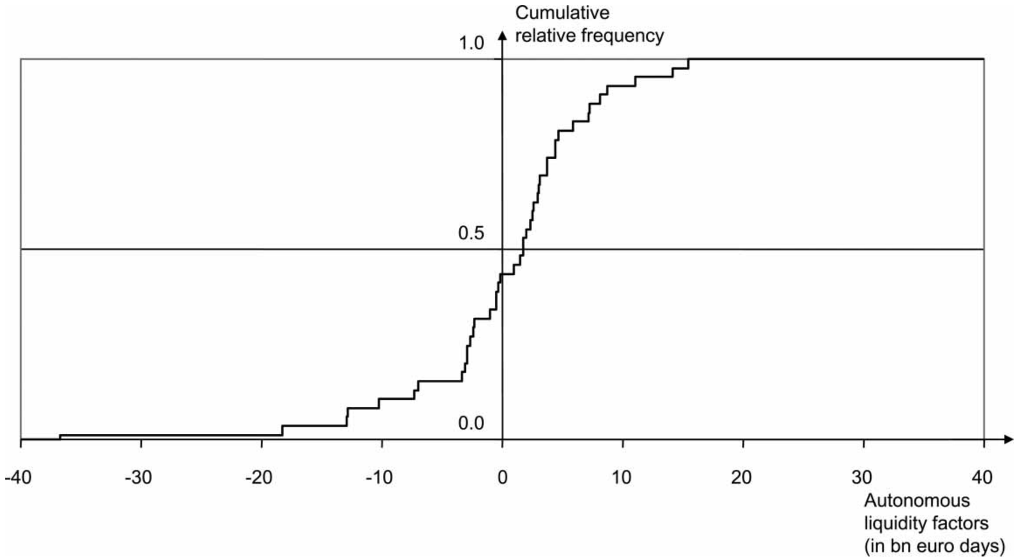

Figure 1. Autonomous factors.
The figure shows the cumulative distribution function of the empirical distribution of unanticipated autonomous liquidity factor shocks (liquidity absorbing) in the euro area between the last operation and the end of the reserve maintenance period. The data set covers the periods ending in the months July 2000 through January 2004.

and lower boundaries of the interest rate corridor are reached at the end of the reserve accounting period.

PROPOSITION 1 (Poole's Lemma) *Let S denote the central bank's aggregate liquidity supply expected for the maintenance period, and assume that the interest rate follows a martingale process. Then the market rate before time t_s is given by*

$$r(S) = (1 - G(S - \bar{R}))r^{\mathrm{L}} + G(S - \bar{R})r^{\mathrm{D}}, \tag{1}$$

where $G(S - \bar{R})$ is the probability of ending the maintenance period with recourse to the deposit facility.

Proof See text above. □

Proposition 1 provides a link between the central bank's supply of liquidity and the market rate prevailing on the interbank market. For example, if the market expects the allotment policy to be restrictive, or else that autonomous factors will drain the banking system after the last operation in a given period, then the market rate will increase above the mid of the corridor. Indeed, the existing experience with corridor systems suggests that this mechanism captures the first-order determinant for short-term money market rates.

Derivation of the liquidity target

Proposition 1 suggests that the monetary authority, in order to implement its interest rate target $r^* \in (r^{\mathrm{D}}; r^{\mathrm{L}})$, would have to provide for average liquidity conditions S^* such that

$$r^* = (1 - G(S^* - \bar{R}))r^{\mathrm{L}} + G(S^* - \bar{R})r^{\mathrm{D}}.$$

Solving for S^* gives the neutral average liquidity as a sum of aggregate reserve requirements and a percentile of the autonomous factor distribution

$$S^* = \bar{R} + G^{-1}\left(\frac{r^{\mathrm{L}} - r^*}{r^{\mathrm{L}} - r^{\mathrm{D}}}\right). \tag{2}$$

For example, in the case where the corridor is symmetric around the target rate, i.e., when $r^* = (r^{\mathrm{L}} + r^{\mathrm{D}})/2$, the liquidity target would be just the sum of reserve requirements and the median of the autonomous factor distribution.

As mentioned in the introduction, we will assume that the central bank is interested in smooth implementation, captured by a target for the total central bank credit L^* outstanding to the banking system. A natural candidate for L^* results if the outstanding central bank credit is held constant over the reserve accounting period, so that we define

$$L^* = \frac{S^*}{T}. \tag{3}$$

Indeed, under this condition, a constant volume L^* of outstanding central bank credit just implements the interest rate target r^*.

3. Benchmark allotments

We consider now in more detail the central bank's allotment decision in the last regular operation of the reserve accounting period $[0;T]$. We envisage a central bank that, like the European Central Bank (ECB), performs regular operations that are spaced on the time axis (not daily), with each operation providing a significant fraction of the overall liquidity supply in the current period. The possibility of additional non-regular open market activities such as fine-tuning will be ignored.

While in principle, there can be several regular operations providing liquidity during the reserve maintenance period, it will be sufficient for our purposes to model explicitly only the last two operations in the period, and to consider an aggregate of the supply through earlier operations (see Figure 2 for illustration).

Specifically, we consider the penultimate operation (henceforth tender A) and the last operation (tender B). Let t_{A} and t_{B} denote the time of tenders A and B, respectively, where $0 < t_{\mathrm{A}} < t_{\mathrm{B}} < t_{\mathrm{s}}$. Moreover, let $X_{\mathrm{A}} \geq 0$ and $X_{\mathrm{B}} \geq 0$ be the allotment volumes in tenders A and B, respectively, and let S_0 denote the liquidity supply for the current reserve maintenance period provided by operations before tender A. For specificity, we will assume that the liquidity allocated in tenders A and B is outstanding over the time intervals $[t_{\mathrm{A}}; T]$ and $[t_{\mathrm{B}}; T]$, respectively, while liquidity provided in earlier operations matures before t_{B}.

Under these assumptions, the expected aggregate liquidity supply at the end of the period is given by

$$S = S(X_{\mathrm{A}}, X_{\mathrm{B}}) = S_0 + (T - t_{\mathrm{A}})X_{\mathrm{A}} + (T - t_{\mathrm{B}})X_{\mathrm{B}}. \tag{4}$$

Replacing S by S^* and subsequently solving for X_{B} yields the allotment that implements the interest rate target r^* as

$$X_{\mathrm{B}}^b(X_{\mathrm{A}}) = \frac{1}{T - t_{\mathrm{B}}}\{S^* - S_0 - (T - t_{\mathrm{A}})X_{\mathrm{A}}\}. \tag{5}$$

This amount will be referred to as the *benchmark allotment* in tender B. With this allotment, the central bank implements the target interest rate r^*.[2]

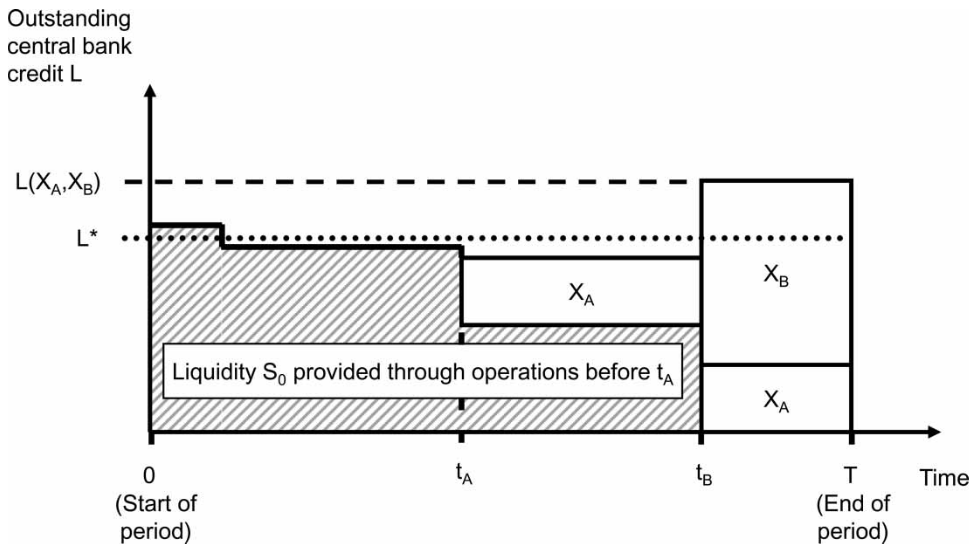

Figure 2. Time structure of liquidity supply.
There are two tenders at times t_A and t_B that provide liquidity to the banking system. The central bank credit allotted in these operations remains outstanding until the end of the reserve accounting period. Operations before t_A are not modeled explicitly, but represented by the liquidity aggregate S_0. Central bank credit allotted in these earlier operations is assumed to mature before t_B.

To determine the allotment necessary to implement L^*, note that from the assumptions on the time structure made above, the outstanding liquidity after tender B is simply the sum of the allotments made in tenders A and B, i.e.,

$$L = L(X_A, X_B) = X_A + X_B. \tag{6}$$

Implementing the liquidity target L^* requires therefore an allotment of the *liquidity refill*:

$$X_B^1(X_A) = L^* - X_A. \tag{7}$$

However, as the next proposition shows, this will be typically inconsistent with implementing the interest rate target.

PROPOSITION 2 *The central bank can implement both the interest rate and the liquidity target simultaneously if and only if*

$$X_A = X_A^b = \frac{1}{t_B - t_A}(t_B L^* - S_0). \tag{8}$$

However, if $X_A < X_A^b$, then $L > L^*$ or $r(S) > r^*$. Moreover, if $X_A > X_A^b$, then $L < L^*$ or $r(S) < r^*$.

Proof The first assertion follows immediately from $X_B^1(X_A) = X^b(X_A)$ using Equations (5), (7), and (3). Consider now the case $X_A < X_A^b$. If $X_B < X_B^b(X_A)$, then $r(S) > r^*$, so that we are done in this case. Assume therefore that $X_\mathrm{B} \geq X_\mathrm{B}^b(X_\mathrm{A})$. Then

$$\begin{aligned} X_\mathrm{A} + X_\mathrm{B} &\geq X_\mathrm{A} + X_\mathrm{B}^b(X_\mathrm{A}) \\ &= \frac{1}{T - t_\mathrm{B}}\{S^* - S_0 - (t_\mathrm{B} - t_\mathrm{A})X_\mathrm{A}\} \\ &= \frac{1}{T - t_\mathrm{B}}\{S^* - S_0 - (t_\mathrm{B} - t_\mathrm{A})X_\mathrm{A}^b\} \\ &= L^*. \end{aligned}$$

Thus, we have $L > L^*$. The case $X_\mathrm{A} > X_\mathrm{A}^b$ can be treated analogously and is therefore omitted. □

We will refer to X_A^b as the benchmark allotment in tender A. In the sequel, we will focus on the case where $X_\mathrm{A} < X_\mathrm{A}^b$, and refer to this case as *underbidding*. Under this condition, the banking system has built up a red position at the time of tender B, i.e., in order to satisfy reserve requirements as an average over the maintenance period, the outstanding central bank credit after tender B must exceed L^* or the interest rate will increase above r^*. Thus, following an event of underbidding, there is a trade-off between liquidity and interest rate smoothing.

4. Optimal allotment

We will impose that the central bank minimizes a weighted sum of quadratic deviations from the interest rate and liquidity targets by choosing an optimal allotment X_B in the last operation. Assuming for simplicity that the central bank's allotment constraint (given by the total of incoming bids) is not binding in tender B, the central bank's problem has the form

$$X_\mathrm{B}^*(X_\mathrm{A}) = \arg\min_{X_\mathrm{B} \geq 0} (r(S) - r^*)^2 + \mu(L - L^*)^2 \tag{9}$$

$$\text{s.t. Equations (1), (4), (3), and (6),}$$

where $\mu > 0$ is the weight assigned by the central bank to the liquidity target. This problem asks for the optimal point on the feasibility curve, i.e., the set of combinations of quadratic interest rate and liquidity deviations that result from feasible allotments in tender B. The involved trade-off is that a lower X_B leads to an increased market rate, while the outstanding liquidity is lowered. The next proposition gives a range for the optimal allotment in tender B, after an occurrence of underbidding in tender A.

PROPOSITION 3 *Assume that $X_\mathrm{A} < X_\mathrm{A}^b$. Then $X_\mathrm{B}^1(X_\mathrm{A}) < X_\mathrm{B}^b(X_\mathrm{A})$, and the optimal allotment $X_\mathrm{B}^*(X_\mathrm{A})$ lies in the open interval $[X_\mathrm{B}^1(X_\mathrm{A}); X_\mathrm{B}^b(X_\mathrm{A})]$.*

Proof By Proposition 2, if $X_\mathrm{A} < X_\mathrm{A}^b$, and $X_\mathrm{A} + X_\mathrm{B} \leq L^*$, we must have that $r(S) > r^*$, and therefore $X_\mathrm{B} < X_\mathrm{B}^b(X_\mathrm{A})$. Replacing X_B by $X_\mathrm{B}^1(X_\mathrm{A})$ yields the first assertion. Consider now the necessary first order conditions for an optimal allotment $X_\mathrm{B}^*(X_\mathrm{A})$, which reads

$$r'(S)(r(S) - r^*)(T - t_\mathrm{B}) + \mu(L - L^*) = 0, \tag{10}$$

where $S = S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A}))$. By Proposition 1, we have $r'(S) < 0$. To provoke a contradiction, assume that $X_\mathrm{B}^*(X_\mathrm{A}) \geq X_\mathrm{B}^b(X_\mathrm{A})$. But then $r(S) \leq r^*$. From Proposition 2, in the case of underbidding, we must therefore have $L = L(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}})) > L^*$. However, this contradicts Equation (10), proving $X_{\mathrm{B}}^*(X_{\mathrm{A}}) < X_{\mathrm{B}}^b(X_{\mathrm{A}})$. Similarly, if $X_{\mathrm{B}}^*(X_{\mathrm{A}}) \leq L^* - X_{\mathrm{A}}$, then $L \leq L^*$. By Proposition 2, this implies $r(S) > r^*$, contradicting Equation (10). This proves $X_{\mathrm{B}}^*(X_{\mathrm{A}}) > X_{\mathrm{B}}^1(X_{\mathrm{A}})$, and thereby the proposition. □

Thus, the optimal allotment lies between two focal allotment sizes. The first is the benchmark amount, i.e., the allotment that ends the maintenance period with an average liquidity position that allows banks to satisfy reserve requirements in a regular way. With this allotment, the interest rate target $r = r^*$ is met, but the outstanding central bank credit will exceed the target L^*. The second focal allotment size is the liquidity refill that matches the target $L = L^*$ for the outstanding central bank credit. This allotment is too small to establish neutral conditions at the end of the period, forcing interest rates to increase above the target r^*.

What is the shape of the feasibility curve between these two points? To answer this question, note that the first-order Taylor expansion of the right-hand side of Equation (1) with respect to S around S^* reads

$$r(S) \approx r^* - \rho(S - S^*),$$

where

$$\rho = G'(S^* - \bar{R})(r^{\mathrm{L}} - r^{\mathrm{D}}) \tag{11}$$

is a measure of the end-of-period liquidity effect for small variations in S. Proposition 3 tells us that $S(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}})) < S^*$ after an occurrence of underbidding. If we lower X_{B} from the benchmark allotment, this increases the interest rate and lowers the outstanding credit. In a neighborhood of the benchmark allotment, the monetary authority therefore faces a linear trade-off between interest rate and liquidity smoothing. Ignoring higher-order effects for the moment, this trade-off translates into a convex feasibility set in the plane of quadratic deviations by a change of variables. In fact, in the neighborhood of the benchmark allotment, the feasibility set has locally the shape of a parabola.

For allotments that are significantly below the benchmark allotment, however, the central bank's lending facility puts an upper bound on the quadratic deviation from the interest rate target. Indeed, since the tails of the distribution of the autonomous liquidity factors must eventually diminish, the feasibility curve, for small and declining values of X_{B}, will bend backwards and become concave. Figure 3 shows the shape of the feasibility curve for an example where the target rate lies in the center of the corridor, i.e., $r^* = (r^{\mathrm{L}} + r^{\mathrm{D}})/2$, the autonomous factor distribution is normal, and the underbidding has been sufficiently pronounced. Clearly, if the underbidding is very mild, so that even the liquidity-neutral allotment would not cause the market rate to approach the marginal lending rate, then the trade-off between liquidity and interest-rate smoothing is convex, and an interior solution is optimal.

The two curves in Figure 3 represent the set of possible combinations of quadratic deviations from the liquidity and interest rate target for two specific sets of initial conditions. The straight lines represent the central bank's indifference curves. Focus for the moment on the left hand curve, marked by the phrase 'weaker underbidding'. It can be seen from the illustration that in this case, the optimal allotment is slightly smaller than the benchmark allotment. However, the feasibility curve is concave for intermediate values of X_{B}, suggesting the possibility of a discontinuous central bank reaction to smoothly changing parameters. In fact, we can show that a discontinuity is quite typical if the liquidity effect is not too small.

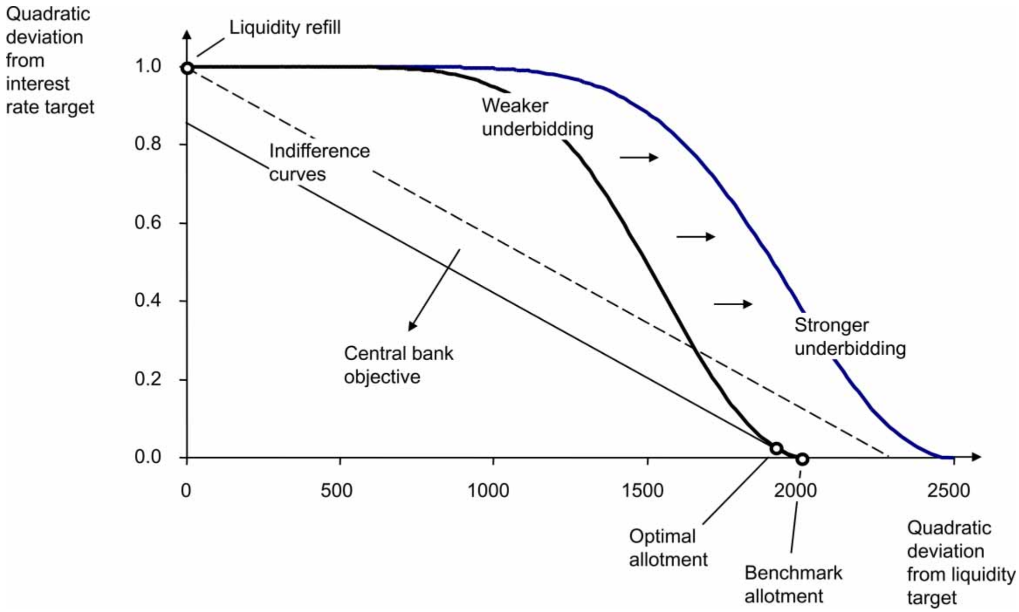

Figure 3. Feasibility following underbidding.
The benchmark allotment implements the interest rate target, but misses the liquidity target. In contrast, the liquidity refill implements the liquidity target, but misses the interest rate target. The optimal allotment is close to the benchmark allotment for weaker underbidding and essentially equal to the liquidity refill for stronger underbidding.

PROPOSITION 4 *Assume that $\mu > 0$ is not too small. Then there is a threshold level $X_A^* \in (0; X_A^b)$, such that for any sufficiently small $\varepsilon > 0$, there is a parameter $\sigma_A^* > 0$, such that for all autonomous factor distributions G(.) with standard deviation $\sigma_A \leq \sigma_A^*$, the optimal allotment $X_B^*(X_A)$ lies in the interval $[X_B^b(X_A) - \varepsilon; X_B^b(X_A)]$ for $X_A > X_A^* + \varepsilon$, and in the interval $[X_B^1(X_A); X_B^1(X_A) + \varepsilon]$ for $X_A < X_A^* - \varepsilon$.*

Proof See Appendix 1. □

The possibility of a discontinuous central bank reaction is illustrated in the second curve that corresponds to a set-up with a slightly more pronounced underbidding. Here, the optimal allotment is very close to the liquidity refill. The rationale for this possibility is that if the volume of tender A has been insufficient, and tender B is close to the end of the period, then the benchmark allotment would be very large. Injecting the benchmark allotment would therefore imply an outstanding central bank credit temporarily much larger than the target. On the other hand, once the allotment is chosen to be tight, there will be essentially no further deviation from the interest rate target by a somewhat tighter allotment. As a consequence, the feasibility curve is concave for intermediate allotment volumes, and it may be optimal to allocate only the amount that aligns the aggregate outstanding central bank credit with the target level.

5. Estimation of the liquidity effect

As a validation of the practical relevance of Proposition 4, we will now estimate the size of the liquidity effect for the case of the euro area. The method of estimation is indirect and relies on results obtained in Section 2. Indeed, by Proposition 1, the impact of a liquidity injection on the market rate is just a linear transformation of the cumulative distribution function $G(.)$ of autonomous liquidity factors. It is therefore sufficient for our purposes to estimate $G(.)$.

The data set used comprises daily realizations of aggregate autonomous liquidity factors in the euro area, as well as certain forecasts on these figures that have been published by the ECB since the end of June 2000. The data has been taken from material made available by the ECB in the forefront of the January 2005 workshop on monetary policy implementation in the euro area. The data covers altogether 43 maintenance periods, the first of which ended by July 23, 2000, and the last of which ended by January 23, 2004.

For every reserve maintenance period $m = 1, \ldots, 43$, we have calculated the unanticipated component of the changes to autonomous liquidity factors, in the sequel referred to as the *autonomous factor shock*, as follows. Let $A(d)$ denote autonomous liquidity factors at calendar date d. Denote by $d^0(m)$ the date of the announcement of the last main refinancing operation in the maintenance period m, and by $d^1(m)$ the date of the last day in period m. Moreover, let $A^{\mathrm{f}}(m)$ denote the forecasted average of the autonomous liquidity factors over the period $d^0(m)$ through $d^1(m)$, as published by the ECB at date $d^0(m)$. Then the autonomous factor shock at the end of period m is defined as

$$B(m) = \sum_{d=d^0(m)}^{d^1(m)} \{A(d) - A^{\mathrm{f}}(m)\}.$$

We found that the mean of the historical distribution of the variable $B(m)$ over the considered period is $\bar{B} = -0.1$ bn euro days, and that its standard deviation is $\sigma_{\mathrm{B}} = 8.8$ bn euro days.

Assuming a normally distributed distribution, the liquidity effect can be quantified via Equation (11) in a one-dimensional figure of $\rho \approx 9.0$ basis points per bn euro days. It should be noted, though, that our estimation is based on the assumption that the liquidity situation is perfectly observable for market participants. In reality, the effect of a liquidity imbalance in the market should be smaller than this figure due to imperfect information about autonomous liquidity factors (Ewerhart et al. 2004). Indeed, using an alternative approach that estimates the spread between the interbank deposit rate index EONIA and the mid of the interest rate corridor as a function of a large vector of observables, Wurtz (2003) finds a value of about half the size of our estimate.

A numerical example

We have used the above estimation result to calculate two feasibility curves numerically (Figure 3). In both scenarios, the parameters have been fixed as follows:

$$r^{\mathrm{D}} = 3.75\% \quad r^* = 4.75\% \quad r^{\mathrm{L}} = 5.75\% \quad \bar{R} = 120$$
$$S_0 = 99 \quad t_{\mathrm{A}} = 0.7 \quad t_{\mathrm{B}} = 0.95 \quad T = 1.$$

The distribution of autonomous factors was assumed to be normal with mean zero and standard deviation $0.3 \approx \sigma_{\mathrm{B}}/30$ days. In the first scenario (‘weaker underbidding’), we assumed $X_{\mathrm{A}} = 51$, in the second (‘stronger underbidding’) $X_{\mathrm{A}} = 50$. As the figure suggests, with weaker underbidding, the optimal allotment is the regular benchmark amount, whereas with stronger underbidding, it is only the liquidity refill. To check the robustness of our predictions with respect to potential estimation errors, we have repeated the computation for a doubled standard deviation (corresponding to a smaller liquidity effect), with no qualitative changes in the results.

6. Extensions

Overlapping operations

The model employed in the formal analysis relies on the assumption that operations do not hang over into the subsequent maintenance period. Taking account of the effects of the allotment policy on liquidity and interest rate smoothing in the subsequent period leads to an infinite-horizon set-up with discounting as expressed by the central bank objective function

$$\hat{U} = -\int_{t_0}^{\infty} \delta^{t-t_0}\{(r_t - r^*)^2 + \mu(L_t - L^*)^2\}\mathrm{d}t,$$

where t_0 is the time of the last tender in the prevailing maintenance period, the parameters r_t and L_t denote the interest rate and the outstanding central bank credit, respectively, at time t, and $\delta \in (0; 1)$ is the discount factor. In the working paper version (Ewerhart et al. 2003), we calculate numerically the optimal intertemporal allotment policy in a set-up with four overlapping operations per period and a linear approximation of the liquidity effect. It turns out that the shape of the feasibility curve does not differ significantly from our prediction in the one-period model.

The intuitive reason for the robustness is that the central bank will optimally reduce much of the imbalance with the first operation in the subsequent reserve maintenance period. As a consequence, the effect of the liquidity imbalance on the interest rate in the subsequent period is very small, and the effect on the outstanding liquidity is essentially restricted to the time before the first operation in the subsequent period. Thus, in a first-order approximation, the infinite-horizon set-up reduces to a one-period problem with a modified weight on the liquidity deviation, as given by

$$\hat{\mu} = \mu\frac{1 - t_{\mathrm{B}}}{1 + t'_{\mathrm{A}} - t_{\mathrm{B}}},$$

where t'_{A} denotes the time of the first open market operation in the subsequent period. As a consequence, the theoretical predictions remain unaffected by considering an infinite-horizon set-up. In fact, in the case of the euro area, the propagation of the liquidity imbalance beyond the second open market operation of the subsequent reserve maintenance period was effectively made impossible by the use of so-called split operations (ECB 2003).

Excess liquidity

A variation of the one-period model occurs if there is excess liquidity before tender B. In the model, the case $X_{\mathrm{A}} > X_{\mathrm{A}}^b$ corresponds to an interpretation where the monetary authority has decided to allot in tender A more than the benchmark amount. Alternatively, a liquidity-providing shock might have occured between tenders A and B. There are again two focal allotment sizes. The first is obviously the benchmark amount $X_{\mathrm{B}}^b(X_{\mathrm{A}})$, which guarantees that the overall liquidity position at the end of the maintenance period is such that the market rate reaches both the marginal lending and the deposit rate with equal probability. The benchmark would typically be small in this scenario.

The second focal allotment $X_{\mathrm{B}}^1(X_{\mathrm{A}})$, which is larger in this scenario, is the one that generates, from the settlement day of tender B onwards, an outstanding central bank credit that corresponds to the target L^*. Allotting the liquidity refill means here to flood the market with liquidity, so that the market rate would drop to the deposit rate. One can show that the shape of the feasibility curve is very similar to the underbidding case. Also in this case, the monetary authority faces a non-convex trade-off between liquidity and interest rate smoothing. Thus, if the imbalance is sufficiently strong, it may become optimal for the central bank to depart from the benchmark in tender B. However, in contrast to the scenario of an undersized tender A, the theoretically optimal allotment may not be feasible due to insufficient demand.

7. Conclusion

On several occasions during the period June 2000 through March 2004, the Eurosystem experienced underbidding in its liquidity-providing open market operations, implying a temporary tightness in the euro money market. While the demand in the subsequent operation was typically very strong, the ECB regularly did not decide to fully alleviate the liquidity shortage, causing short-term interest rates to rise significantly above the main policy rate. In this article, we have derived the optimal allotment in response to underbidding in a model that captures some of the institutional features of the Eurosystem's operational framework for monetary policy implementation.

The formal analysis suggests that, somewhat surprisingly, creating a substantial monetary tightness at the end of the maintenance period after an occurrence of underbidding may be consistent with pursuing an optimal allotment policy. In fact, we showed that there is a threshold band for the accumulated aggregate liquidity position in the banking system at the last open market operation in a given maintenance period, so that the benchmark allotment is optimal whenever liquidity conditions are above the threshold, and a tight allotment is optimal whenever liquidity conditions are below the threshold. This provides a possible explanation for the Eurosystem's recurring and significant deviations from the benchmark allotment rule following occurrences of significant underbidding.[3]

The analysis may also provide a rationale for changes of the operational framework implemented by the ECB in March 2004. The new framework relies on non-overlapping transactions with a maturity of one week only. Transactions also do not hang over into the subsequent period, and the Governing Council confines itself to making policy decisions only at the beginning of maintenance periods. With the new scheme, interest rate expectations should not affect the overnight rate in the current maintenance period, so that underbidding should be much less likely under the new regime. In this sense, the recently implemented changes to the operational framework would make the discontinuous reaction to unbalanced liquidity conditions less often necessary and would therefore allow commercial banks to satisfy their reserve requirements in a smoother way than before March 2004.[4]

Notes

1. The opinions expressed in this article are those of the authors alone, and do not necessarily reflect the views of the European Central Bank, Danmarks Nationalbank, or the Banque de France. This article has been drafted while the first-named author was visiting the Monetary Policy Stance Division of the European Central Bank in fall 2002. The analysis has benefited from comments received in seminars at the European Central Bank, at the University of Zurich, at the Annual Meeting of the German Economic Society, at the Tinbergen Institute of the University of Rotterdam, at the SAET conference on the Island of Rhodes, at the CfS Workshop 'Implementing Monetary Policy' in Frankfurt, and at the University College London.
2. Indeed, Equation (5) is a formal counterpart of the Eurosystem's definition (ECB 2002).
3. An alternative explanation for the ECB's reluctance to bail out the banking system after an occurrence of underbidding is that the increased rates at the end of the period should make underbidding unprofitable. However, this argument should properly be valid only for the initial episodes of underbidding. It was realized soon that, unexpectedly, the two-week swap rate continued to fall below the minimum bid rate on the day of a critical operation, despite the substantial threat (ECB 2003, Table I). Proposition 4 suggests an explanation that is independent of reputation effects.

4. It is fair to say that the liquidity turmoil that began in August 2007 and has not come to an end at the time of final changes to this manuscript (November 2008) has influenced our views on optimal central bank liquidity policy. The market environment during the turmoil has so far been one of credit rationing and moral hazard. This invalidates the intuition underlying Poole's Lemma, but not the trade-off between liquidity and interest-rate smoothing. Indeed, the first thing to note is that with frictions in the money market, the central bank's liquidity target may increase strongly above the level identified in Section 2 for the case of a frictionless market. This is because, with frictions, commercial banks may not have the incentives to exchange liquidity as quickly as needed, so a much larger stock of liquidity in the market is necessary. At the same time, and this is the second thing to note, the objective function of the central bank changes. The liquidity situation becomes much more important for the central bank during the turmoil than the expected interest rate at the final day of the maintenance period. This does not say that central bankers are not concerned about interbank rates, to the contrary. But during a turmoil, there is less weight being put by policy makers on the relative likelihood of reaching the upper or lower end of the corridor on the last day of the maintenance period, compared to achieving a sufficiently generous liquidity provision during the maintenance period. Introducing these two changes to our model, i.e., a larger L^* and a larger μ, indifference curves of the central bank become steeper, so it will often be optimal to choose the liquidity refill. These considerations suggest that the trade-off between liquidity and interest-rate smoothing is relevant also for a market under stress.

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Appendix 1: proof of Proposition 4

In the sequel, we will use the notation

$$D(X_{\mathrm{A}}, X_{\mathrm{B}}) = (r(S(X_{\mathrm{A}}, X_{\mathrm{B}})) - r^*)^2 + \mu(L(X_{\mathrm{A}}, X_{\mathrm{B}}) - L^*)^2$$

for the weighted sum of the deviations from the central bank's interest rate and liquidity targets. The proof proceeds in six steps.

Step 1. Choose X_{A}^* such that

$$D(X_{\mathrm{A}}^*, X_{\mathrm{B}}^b(X_{\mathrm{A}}^*)) = (r^{\mathrm{L}} - r^*)^2. \tag{12}$$

In Figure 3, this allotment corresponds to the intersection of the abscissa with an indifference curve through the point $[0; (r^{\mathrm{L}} - r^*)^2]$, which is located just above the point corresponding to the liquidity refill. We claim that, if μ is not too small, then such an $X_{\mathrm{A}}^* \in (0; X_{\mathrm{A}}^b)$ exists. To see why, we consider allotments $X_{\mathrm{A}} = 0$ and $X_{\mathrm{A}} = X_{\mathrm{A}}^b$ in the sequel. Note that

$$D(0, X_{\mathrm{B}}^b(0)) = \mu(X_{\mathrm{B}}^b(0) - L^*)^2.$$

Thus, if

$$\mu > \mu^* = \frac{(r^{\mathrm{L}} - r^*)^2}{[X_{\mathrm{B}}^b(0) - L^*]^2},$$

one obtains

$$D(0, X_{\mathrm{B}}^b(0)) > (r^{\mathrm{L}} - r^*)^2.$$

Clearly, the parameter μ^* is finite because $X_{\mathrm{B}}^b(0) > L^*$ by Proposition 2. On the other hand,

$$D(X_{\mathrm{A}}^b, X_{\mathrm{B}}^b(X_{\mathrm{A}}^b)) = 0 < (r^{\mathrm{L}} - r^*)^2.$$

Hence, by the continuity of G(.), and by the intermediate value theorem, if μ is not too small, there exists an $X_{\mathrm{A}}^* \in (0; X_{\mathrm{A}}^b)$ satisfying Equation (12).

Step 2. Let $\varepsilon \in (0; X_{\mathrm{A}}^*)$. Assume that $X_{\mathrm{A}} < X_{\mathrm{A}}^* - \varepsilon$. By Proposition 3, we know that $X_{\mathrm{B}}^*(X_{\mathrm{A}}) > X_{\mathrm{B}}^1(X_{\mathrm{A}})$. We have to show that $X_{\mathrm{B}}^*(X_{\mathrm{A}}) < X_{\mathrm{B}}^1(X_{\mathrm{A}}) + \varepsilon$. To provoke a contradiction, we assume

$$X_{\mathrm{B}}^*(X_{\mathrm{A}}) \geq X_{\mathrm{B}}^1(X_{\mathrm{A}}) + \varepsilon. \tag{13}$$

The idea will be to show that under condition (13), the allotment $X_{\mathrm{B}}^*(X_{\mathrm{A}})$ would be dominated by the benchmark allotment. Note first that from condition (13),

$$L(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}})) - L^* \geq \varepsilon.$$

Thus, using the optimality of $X_{\mathrm{B}}^*(X_{\mathrm{A}})$, we have

$$\begin{aligned} [r(S(X_{\mathrm{A}}, X_{\mathrm{B}}^1(X_{\mathrm{A}}))) - r^*]^2 &= D(X_{\mathrm{A}}, X_{\mathrm{B}}^1(X_{\mathrm{A}})) \\ &\geq D(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}})) \\ &\geq [r(S(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}}))) - r^*]^2 + \mu\varepsilon^2 \end{aligned}$$

This implies

$$[r(S(X_{\mathrm{A}}, X_{\mathrm{B}}^*(X_{\mathrm{A}}))) - r^*]^2 \leq (r^{\mathrm{L}} - r^*)^2 - \mu\varepsilon^2. \tag{14}$$

In step 5 of the proof, we will determine a value $\sigma_{\mathrm{A}}^* > 0$ such that Equation (14) cannot hold for distributions $G(.)$ with a standard deviation $\sigma_{\mathrm{A}} \leq \sigma_{\mathrm{A}}^*$.

Step 3. We continue to consider the case $X_{\mathrm{A}} < X_{\mathrm{A}}^* - \varepsilon$. We show first that

$$X_{\mathrm{A}}^* + X_{\mathrm{B}}^b(X_{\mathrm{A}}^*) \geq X_{\mathrm{A}} + X_{\mathrm{B}}^*(X_{\mathrm{A}}). \tag{15}$$

Indeed, otherwise we had by Proposition 2 that

$$0 < X_{\mathrm{A}}^* + X_{\mathrm{B}}^b(X_{\mathrm{A}}^*) - L^* < X_{\mathrm{A}} + X_{\mathrm{B}}^*(X_{\mathrm{A}}) - L^*. \tag{16}$$

On the other hand, we have

$$S^* > S(X_{\mathrm{A}}^*, X_{\mathrm{B}}^1(X_{\mathrm{A}}^*)) > S(X_{\mathrm{A}}, X_{\mathrm{B}}^1(X_{\mathrm{A}})),$$

and therefore

$$0 < r(S(X_{\mathrm{A}}^*, X_{\mathrm{B}}^1(X_{\mathrm{A}}^*))) - r^* < r(S(X_{\mathrm{A}}, X_{\mathrm{B}}^1(X_{\mathrm{A}}))) - r^*. \tag{17}$$

Inequalities (16) and (17) imply that

$$\begin{aligned} D(X_A, X_B^*(X_A)) &\geq \mu(X_A + X_B^*(X_A) - L^*)^2 \\ &> \mu(X_A^* + X_B^b(X_A^*) - L^*)^2 \\ &= (r^L - r^*)^2 \\ &> (r(S(X_A, X_B^1(X_A))) - r^*)^2 \\ &= D(X_A, X_B^1(X_A)), \end{aligned}$$

contradicting the optimality of $X_B^*(X_A)$. This proves Equation (15), and therefore

$$X_B^*(X_A) \leq X_A^* - X_A + X_B^b(X_A^*).$$

Hence, taking account of $S(X_A, X_B^b(X_A)) = S^*$, we find that

$$S(X_A, X_B^*(X_A)) \leq S^* - (t_B - t_A)(X_A^* - X_A) < S^* - (t_B - t_A)\varepsilon.$$

Using Equation (2) yields

$$S(X_A, X_B^*(X_A)) - \bar{R} < G^{-1}\left(\frac{r^L - r^*}{r^L - r^D}\right) - (t_B - t_A)\varepsilon. \tag{18}$$

In order to proceed, we will need an auxiliary result from probability theory which we state and prove in the subsequent step for lack of a suitable reference.

Step 4. Let $\alpha \in (0; 1)$ and $\beta > 0$ such that

$$\sigma_A < \beta\sqrt{1-\alpha}. \tag{19}$$

We claim then that

$$pr\left\{\tilde{A} \leq G^{-1}(\alpha) - \beta\right\} \leq \frac{\sigma_A^2}{(\beta - \sigma_A)/\sqrt{1-\alpha}}, \tag{20}$$

which will be seen to be a one-sided inequality of the Chebyshev type for deviations from the α percentile. To prove the claim, note that from Chebyshev's inequality [see, e.g., Loeve (1963)],

$$pr\left\{|\tilde{A} - E[\tilde{A}]| \geq \frac{\sigma_A}{\sqrt{1-\alpha}}\right\} \leq 1 - \alpha.$$

But then also

$$pr\left\{\tilde{A} \geq E[\tilde{A}] + \frac{\sigma_A}{\sqrt{1-\alpha}}\right\} \leq 1 - \alpha.$$

Thus, an upper bound for the α percentile relative to the mean can be given by

$$G^{-1}(\alpha) \leq E[\tilde{A}] + \frac{\sigma_A}{\sqrt{1-\alpha}}.$$

Applied to the above problem, this implies

$$\begin{aligned} pr\{\tilde{A} \leq G^{-1}(\alpha) - \beta\} &\leq pr\left\{\tilde{A} - E[\tilde{A}] \leq \frac{\sigma_A}{\sqrt{1-\alpha}} - \beta\right\} \\ &\leq pr\left\{|\tilde{A} - E[\tilde{A}]| \leq \beta - \frac{\sigma_A}{\sqrt{1-\alpha}}\right\}, \end{aligned}$$

where we have used assumption (19). Applying Chebychev's inequality another time proves the claim.

Step 5. Combining Equations (18) and (20), we get

$$0 < G(S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A})) - \bar{R}) < \frac{\sigma_\mathrm{A}^2}{\beta - \sigma_\mathrm{A}/\sqrt{1-\alpha}}, \tag{21}$$

where $\alpha = (r^\mathrm{L} - r^*)/(r^\mathrm{L} - r^\mathrm{D})$ and $\beta = (t_\mathrm{B} - t_\mathrm{A})\varepsilon > 0$. Thus, if $\sigma_\mathrm{A} \to 0$, then also the right-hand side of Equation (21) goes to zero. From

$$(r(S) - r^*) = (r^\mathrm{L} - r^\mathrm{D})(1 - \alpha^{-1} G(S - \bar{R})),$$

we see that

$$[r(S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A}))) - r^*]^2 \to (r^\mathrm{L} - r^*)^2$$

for $\sigma_\mathrm{A} \to 0$. Thus, there exists $\sigma_\mathrm{A}^* > 0$ such that for all distributions given by $G(.)$ with standard deviation $\sigma_\mathrm{A} \le \sigma_\mathrm{A}^*$, we have that Equation (14) is not satisfied. For these distributions, condition (13) implies a contradiction, so that we have shown the first part of the proposition.

Step 6. Consider now allotments $X_\mathrm{A} > X_\mathrm{A}^* + \varepsilon$. We wish to show that $X_\mathrm{B}^*(X_\mathrm{A}) > X_\mathrm{B}^b(X_\mathrm{A}) - \varepsilon$. As above, to provoke a contradiction, we assume

$$X_\mathrm{B}^*(X_\mathrm{A}) \le X_\mathrm{B}^b(X_\mathrm{A}) - \varepsilon. \tag{22}$$

The idea is to show that in this case, the allotment $X_\mathrm{B}^*(X_\mathrm{A})$ would be dominated by the benchmark allotment. Indeed, from Equations (22) and (2),

$$\begin{aligned} S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A})) - \bar{R} &\le S^* - \bar{R} - (T - t_\mathrm{B})\varepsilon \\ &= G^{-1}(\alpha) - (T - t_\mathrm{B})\varepsilon. \end{aligned}$$

Applying inequality (20), we find

$$G(S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A})) - \bar{R}) \le \frac{\sigma_\mathrm{A}^2}{(\beta' - \sigma_\mathrm{A})/\sqrt{1-\alpha}},$$

where $\beta' = (T - t_\mathrm{B})\varepsilon$. As in Step 5, this implies that

$$D(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A})) \ge [r(S(X_\mathrm{A}, X_\mathrm{B}^*(X_\mathrm{A}))) - r^*]^2 \to (r^\mathrm{L} - r^*)^2 \tag{23}$$

for $\sigma_\mathrm{A} \to 0$. On the other hand,

$$X_\mathrm{A} + X_\mathrm{B}^b(X_\mathrm{A}) < X_\mathrm{A}^* + X_\mathrm{B}^b(X_\mathrm{A}^*) - \frac{t_\mathrm{B} - t_\mathrm{A}}{T - t_\mathrm{B}}\varepsilon,$$

so that

$$\begin{aligned} D(X_\mathrm{A}, X_\mathrm{B}^b(X_\mathrm{A})) &< \mu\left(X_\mathrm{A}^* + X_\mathrm{B}^b(X_\mathrm{A}^*) - L^* - \frac{t_\mathrm{B} - t_\mathrm{A}}{T - t_\mathrm{B}}\varepsilon\right)^2 \\ &= \left(r^\mathrm{L} - r^* - \sqrt{\mu}\frac{t_\mathrm{B} - t_\mathrm{A}}{T - t_\mathrm{B}}\varepsilon\right)^2 \\ &< (r^\mathrm{L} - r^*)^2, \end{aligned}$$

contradicting Equation (23), given that $X_\mathrm{B}^*(X_\mathrm{A})$ has been assumed to be optimal. This proves the second part of the proposition.